

EP MODULAR OPERATORS AND THEIR PRODUCTS

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ABSTRACT. We study first EP modular operators on Hilbert C^* -modules and then we provide necessary and sufficient conditions for the product of two EP modular operators to be EP. These enable us to extend some results of Koliha [*Studia Math.* **139** (2000), 81–90.] for an arbitrary C^* -algebra and the C^* -algebras of compact operators.

1. INTRODUCTION.

A bounded linear operator T with closed range on a complex Hilbert space H is called an EP operator if T and T^* have the same range. This was introduced for matrices by Schwerdtfeger in [18] and has been studied in detail by several authors, see e.g. [3, 4, 6, 7, 13, 15, 17] and references therein. A problem that has been open for over twenty-five years is when the product of two EP matrices is again EP [2]. Hartwig and Katz [11], and Koliha [12] gave necessary and sufficient conditions for a product of two $n \times n$ complex EP matrices to be EP. Djordjević [5] provided a generalization of the result for EP operators on Hilbert spaces. In this note we investigate about the EP operators on Hilbert C^* -modules over an arbitrary C^* -algebra of coefficients, and then we reformulate some results of [12, 13] for the product of EP modular operators.

Since the finite-dimensional spaces, Hilbert spaces and C^* -algebras can all be regarded as Hilbert C^* -modules, one can study EP modular operators in a unified way in the framework of Hilbert C^* -modules. Indeed, a Hilbert C^* -module is an object like a Hilbert space except that the inner product is not scalar-valued, but takes its values in a C^* -algebra of coefficients. Since the geometry of these modules emerges from the C^* -valued inner product, some basic properties of Hilbert spaces like Pythagoras' equality, self-duality, and decomposition into orthogonal complements must be given up. These modules play an important role in the modern theory of C^* -algebras and the study of locally compact quantum groups. A (right) *pre-Hilbert C^* -module* over a C^* -algebra \mathcal{A} is a right \mathcal{A} -module X endowed with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathcal{A}$, $(x, y) \mapsto \langle x, y \rangle$ which is linear in the second variable y

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(and conjugate-linear in x), satisfying the conditions

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, ya \rangle = \langle x, y \rangle a \text{ for all } a \in \mathcal{A},$$

$$\langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0.$$

A pre-Hilbert \mathcal{A} -module X is called a *Hilbert \mathcal{A} -module* if X is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. If X, Y are two Hilbert \mathcal{A} -modules then the set of all ordered pairs of elements $X \oplus Y$ from X and Y is a Hilbert \mathcal{A} -module with respect to the \mathcal{A} -valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$. It is called the *direct orthogonal sum of X and Y* . If V is a (possibly non-closed) \mathcal{A} -submodule of X , then $V^\perp := \{y \in X : \langle x, y \rangle = 0 \text{ for all } x \in V\}$ is a closed \mathcal{A} -submodule of X and $\overline{V} \subseteq V^{\perp\perp}$. A Hilbert \mathcal{A} -submodule V of a Hilbert \mathcal{A} -module X is *orthogonally complemented* if V and its orthogonal complement V^\perp yield $X = V \oplus V^\perp$, in this case, V and its biorthogonal complement $V^{\perp\perp}$ coincide. For the basic theory of Hilbert C*-modules we refer to the books [14, 16].

Throughout the present paper we assume \mathcal{A} to be an arbitrary C*-algebra (i.e. not necessarily unital). We use the notations $Ker(\cdot)$ and $Ran(\cdot)$ for kernel and range of operators, respectively. We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded adjointable operators between X and Y , i.e., all bounded \mathcal{A} -linear maps $T : X \rightarrow Y$ such that there exists $T^* : Y \rightarrow X$ with the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in X, y \in Y$. The C*-algebra $\mathcal{L}(X, X)$ is abbreviated by $\mathcal{L}(X)$.

In this paper we first briefly investigate some basic facts about EP modular operators with closed ranges and then we give some factorizations and characterizations of such operators. If T, S and TS are EP modular operators with closed ranges then $Ran(TS) = Ran(T) \cap Ran(S)$. If, in addition, $Ker(T) + Ker(S)$ is dense in its biorthogonal complement then we obtain $Ker(TS) = \overline{Ker(T) + Ker(S)}$. Some special cases for EP elements of C*-algebras and C*-algebras of compact operators are considered.

2. PRELIMINARIES

Closed submodules of Hilbert modules need not to be orthogonally complemented at all, but Lance states in [14, Theorem 3.2] under which conditions closed submodules may be orthogonally complemented. Let X be a Hilbert \mathcal{A} -module and suppose that an operator T in $\mathcal{L}(X)$ has closed range, then one has:

- $Ker(T)$ is orthogonally complemented in X , with complement $Ran(T^*)$,
- $Ran(T)$ is orthogonally complemented in X , with complement $Ker(T^*)$,
- the map $T^* \in \mathcal{L}(X)$ has closed range, too.

The following results express when the product of two modular operators with closed range again has closed range. Suppose $T, S \in \mathcal{L}(X)$ are bounded adjointable operators with closed range. Then TS has closed range, if and only if $\text{Ker}(T) + \text{Ran}(S)$ is an orthogonal summand in X if and only if $\text{Ker}(S^*) + \text{Ran}(T^*)$ is an orthogonal summand in X . For the proof of the results and historical notes about the problem we refer to [20] and references therein.

Let $T \in \mathcal{L}(X)$, then a bounded adjointable operator $T^\dagger \in \mathcal{L}(X)$ is called the *Moore-Penrose inverse* of T if

$$(2.1) \quad TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T.$$

The notation T^\dagger is reserved to denote the Moore-Penrose inverse of T . These properties imply that T^\dagger is unique and $T^\dagger T$ and TT^\dagger are orthogonal projections. Moreover, $\text{Ran}(T^\dagger) = \text{Ran}(T^\dagger T)$, $\text{Ran}(T) = \text{Ran}(TT^\dagger)$, $\text{Ker}(T) = \text{Ker}(T^\dagger T)$ and $\text{Ker}(T^\dagger) = \text{Ker}(TT^\dagger)$ which lead us to $X = \text{Ker}(T^\dagger T) \oplus \text{Ran}(T^\dagger T) = \text{Ker}(T) \oplus \text{Ran}(T^\dagger)$ and $X = \text{Ker}(T^\dagger) \oplus \text{Ran}(T)$. If T^\dagger exists then $T^\dagger = \lim_{\omega \rightarrow 0^+} (\omega 1 + T^* T)^{-1} T^* = \lim_{\omega \rightarrow 0^+} T(\omega 1 + T^* T)^{-1}$, cf. [19, 21].

Xu and Sheng in [23] have shown that a bounded adjointable operator between two Hilbert C^* -modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore-Penrose, see [10, 19, 21] for more detailed information.

Definition 2.1. Let X be a Hilbert \mathcal{A} -modules. An operator $T \in \mathcal{L}(X)$ is called *EP* if $\text{Ran}(T)$ and $\text{Ran}(T^*)$ have the same closure.

In the Hilbert C^* -module context, one needs to add the extra condition, closeness of the range, in order to get a reasonably good theory. This ensure that an EP operator has a bounded adjointable Moore-Penrose inverse. Like the general theory of Hilbert spaces one can easily see that the following conditions are equivalent:

- T is EP with closed range,
- T and T^* have the same kernel,
- T is Moore-Penrose invertible and $TT^\dagger = T^\dagger T$,
- $\text{Ran}(T)$ is orthogonally complemented in X , with complement $\text{Ker}(T)$.

Proposition 2.2. Let X be a Hilbert \mathcal{A} -module and $T \in \mathcal{L}(X)$ has a closed range. Then the following conditions are equivalent:

- (i) T is EP with closed range,
- (ii) there exists an isomorphism $V \in \mathcal{L}(X)$ such that $T^* = VT$,
- (iii) there exists an isomorphism $V \in \mathcal{L}(X)$ such that $T^\dagger = VT = TV$.

Proof. Suppose T is EP. Then $\text{Ker}(T)$ is orthogonally complemented and there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in \text{Ker}(T)^\perp$, cf. [9, Proposition 1.3]. The latter inequality implies that the module map $T|_{\text{Ker}(T)^\perp} : \text{Ran}(T^*) = \text{Ran}(T) \rightarrow \text{Ran}(T)$ has a bounded inverse, which allows us to define \mathcal{A} -module map

$$Vx = \begin{cases} T^* (T|_{\text{Ker}(T)^\perp})^{-1}x & \text{if } x \in \text{Ran}(T) \\ x & \text{if } x \in \text{Ker}(T). \end{cases}$$

Then $V \in \mathcal{L}(X)$ is an isomorphism which satisfies $T^* = VT$, that is, (ii) holds. To prove (iii) we define

$$Vx = \begin{cases} (T|_{\text{Ker}(T)^\perp})^{-2}x & \text{if } x \in \text{Ran}(T) \\ x & \text{if } x \in \text{Ker}(T), \end{cases}$$

and

$$Sx = \begin{cases} (T|_{\text{Ker}(T)^\perp})^{-1}x & \text{if } x \in \text{Ran}(T) \\ 0 & \text{if } x \in \text{Ker}(T). \end{cases}$$

Then $V, S \in \mathcal{L}(X)$. Using the orthogonal direct sum decompositions, we have $TV = VT = S$. Moreover, T and S satisfy equations (2.1), i.e., S is the Moore-Penrose inverse of T . The remaining parts follow trivially. \square

3. ON THE PRODUCT OF EP MODULAR OPERATORS

In this section we try to generalize some results of Koliha [12, 13] to the framework of Hilbert C*-modules. Some special cases for EP elements of C*-algebras and the C*-algebra of compact operators are also obtained.

Lemma 3.1. *Suppose X is a Hilbert \mathcal{A} -module. Let $T \in \mathcal{L}(X)$ has closed range and $S \in \mathcal{L}(X)$ is an arbitrary operator which commutes with T . Then S commutes with T^\dagger .*

Proof. Using Fuglede-Putnam Theorem [22, Theorem 2.8], the operator S commutes with T^* . Hence, S commutes with $(\omega 1 + T^*T)^{-1}T^*$, $\omega > 0$, and its limit T^\dagger . \square

Proposition 3.2. *Suppose X is a Hilbert \mathcal{A} -module. Let $T, S \in \mathcal{L}(X)$ are EP operators with closed range and $TS = ST$. Then TS is an EP operator with closed range.*

Proof. The operators T, S, T^\dagger and S^\dagger are mutually commute by Lemma 3.1. Therefore TS and $S^\dagger T^\dagger$ satisfies equations (2.1), i.e., TS is Moore-Penrose invertible and its Moore-Penrose

inverse equals $S^\dagger T^\dagger = T^\dagger S^\dagger$. These implies that TS has a closed range and $(TS)^\dagger$ commutes with TS . \square

Proposition 3.3. *Let X be a Hilbert \mathcal{A} -module. If $T, S, TS \in \mathcal{L}(X)$ are EP operators then $T(\overline{\text{Ran}(S)}) \subseteq \overline{\text{Ran}(S)}$ and $S^*(\overline{\text{Ran}(T)}) \subseteq \overline{\text{Ran}(T)}$.*

Proof. Continuity of T implies that $T(\overline{\text{Ran}(S)}) \subseteq \overline{T(\text{Ran}(S))}$ and so $\overline{T(\overline{\text{Ran}(S)})} = \overline{T(\text{Ran}(S))}$. We therefore have

$$T(\overline{\text{Ran}(S)}) \subseteq \overline{T(\overline{\text{Ran}(S)})} = \overline{T(\text{Ran}(S))} = \overline{\text{Ran}(TS)} = \overline{\text{Ran}(S^*T^*)} \subseteq \overline{\text{Ran}(S)}.$$

The second inclusion follows in a similar manner. \square

Koliha in [12, 13] demonstrated that the converse of the above statement is true in the case of finite dimensional spaces. Indeed, he showed that for matrices T and S , TS is EP if and only if $\text{Ker}(T) \subseteq \text{Ker}(TS)$ and $\text{Ran}(TS) \subseteq \text{Ran}(S)$. However, the statement does not hold in the case of Hilbert space or Hilbert C^* -modules.

Example 3.4. let \mathcal{A} be unital C^* -algebra and $H_{\mathcal{A}}$ be the standard Hilbert \mathcal{A} -module which is countably generated by orthonormal basis $\xi_j = (0, \dots, 0, 1, 0, \dots, 0)$, $j \in \mathbb{N}$. Let $W = \overline{\text{span}\{\xi_{2j} : j \in \mathbb{N}\}}$ and S be the orthogonal projection onto the closed submodule W . We define T_0 by $T_0(\xi_1) = \xi_2$, $T_0(\xi_{2j}) = \xi_{2j+2}$ and $T_0(\xi_{2j+1}) = \xi_{2j-1}$, for all $j \in \mathbb{N}$. The inverse of T_0 is defined by $T_0^{-1}(\xi_2) = \xi_1$, $T_0^{-1}(\xi_{2j}) = \xi_{2j-2}$ and $T_0^{-1}(\xi_{2j+1}) = \xi_{2j+3}$. Then T_0 and T_0^{-1} can be extended uniquely to T and T^{-1} on $H_{\mathcal{A}}$ which satisfy $T^* = T^{-1}$. One can easily see that $T(\overline{\text{Ran}(S)}) \subseteq \overline{\text{Ran}(S)}$ and $S^*(\overline{\text{Ran}(T)}) = \text{Ran}(S^*) = \text{Ran}(S) = W \subseteq \overline{\text{Ran}(T)}$. However, TS is not EP since ξ_2 is orthogonal to $\overline{\text{Ran}(TS)}$ and to $\text{Ker}(TS)$.

Lemma 3.5. *Let X and Y be a Hilbert \mathcal{A} -modules and $T \in \mathcal{L}(X, Y)$ has a closed range. If \mathcal{A} -submodule W is orthogonally complemented in Y then the operator T has a matrix representation with respect to the orthogonal sums $X = \text{Ran}(T^*) \oplus \text{Ker}(T)$ and $Y = W \oplus W^\perp$ as follows:*

$$(3.1) \quad T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} W \\ W^\perp \end{bmatrix}.$$

In this case, $A = T_1^* T_1 + T_2^* T_2 : \text{Ran}(T^*) \rightarrow \text{Ran}(T^*)$ is invertible. Moreover,

$$(3.2) \quad T^\dagger = \begin{bmatrix} A^{-1} T_1^* & A^{-1} T_2^* \\ 0 & 0 \end{bmatrix}.$$

Proof. The operator T in $\mathcal{L}(\text{Ran}(T^*) \oplus \text{Ker}(T), W \oplus W^\perp)$ can be represented by the matrix

$$(3.3) \quad T = \begin{bmatrix} T_1 & T_3 \\ T_2 & T_4 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T^*) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} W \\ W^\perp \end{bmatrix},$$

in which,

$$\begin{aligned} T_1 &= T|_{\text{Ran}(T^*)} : \text{Ran}(T^*) \rightarrow W, \\ T_2 &= T|_{\text{Ran}(T^*)} : \text{Ran}(T^*) \rightarrow W^\perp, \\ T_3 &= T|_{\text{Ker}(T)} : \text{Ker}(T) \rightarrow W, \\ T_4 &= T|_{\text{Ker}(T)} : \text{Ker}(T) \rightarrow W^\perp. \end{aligned}$$

Since $T(\text{Ker}(T)) = 0$, we obtain $T_3 = T_4 = 0$. Suppose $x \in \text{Ran}(T^*)$ and $Ax = 0$ then $0 = \langle T_1^* T_1 + T_2 T_2^* x, x \rangle = \langle T_1 x, T_1 x \rangle + \langle T_2 x, T_2 x \rangle \geq \langle T_1 x, T_1 x \rangle \geq 0$, which implies $T_1 x = 0$. Similarly, $T_2 x = 0$ and so $x \in \text{Ker}(T) \cap \text{Ran}(T^*) = \{0\}$, i.e., A is injective. Using closeness of the range of T^* and [20, Lemma 2.1], we have $\text{Ran}(T^* T) = \text{Ran}(T^*)$ which follows subjectivity of A . Hence, A is invertible. Finally, the operators $T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix}$ and $\begin{bmatrix} A^{-1} T_1^* & A^{-1} T_2^* \\ 0 & 0 \end{bmatrix}$ satisfy equations (2.1) which obtain the matrix form for the Moore-Penrose inverse of T . \square

Lemma 3.6. *Let X be a Hilbert \mathcal{A} -module and $T \in \mathcal{L}(X)$ with closed range. Then T is EP if and only if it is of the matrix form*

$$(3.4) \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T) \end{bmatrix},$$

for some invertible operator $T_1 \in \mathcal{L}(\text{Ran}(T), \text{Ran}(T))$.

Proof. Let T be EP then the bounded adjointable $T_1 : \text{Ker}(T)^\perp = \text{Ran}(T) \rightarrow \text{Ran}(T)$, $T_1 x = Tx$, has a bounded inverse. This fact together with the matrix representation (3.1) give us the desired representation.

Conversely, let $T_1 \in \mathcal{L}(\text{Ran}(T), \text{Ran}(T))$ and T admit the matrix representation (3.4). Then

$$(3.5) \quad T^\dagger = \begin{bmatrix} (T_1^* T_1)^{-1} T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is the Moore-Penrose inverse of T which commutes with T , i.e., T is EP. \square

Suppose M and N are submodule of a Hilbert C^* -module X , then $(M + N)^\perp = M^\perp \cap N^\perp$. In particular, if $M + N$ is dense in its biorthogonal complement then

$$(M^\perp \cap N^\perp)^\perp = (M + N)^{\perp\perp} = \overline{M + N}.$$

Theorem 3.7. *Let X be a Hilbert \mathcal{A} -module and $T, S \in \mathcal{L}(X)$ are EP operators with closed ranges. Among the following four properties of T , S and TS , the implication (i) \rightarrow (iii) holds. Moreover, (i) and (ii) are equivalent to (iii) and (iv).*

- (i) TS is an EP operator with closed range.
- (ii) $\text{Ker}(T) + \text{Ker}(S)$ is dense in its biorthogonal complement.
- (iii) $\text{Ran}(TS) = \text{Ran}(T) \cap \text{Ran}(S)$.
- (iv) $\text{Ker}(TS) = \overline{\text{Ker}(T) + \text{Ker}(S)}$.

Proof. Suppose T , S and TS are EP operators with closed ranges. Using Lemmata 3.5, 3.6 and the orthogonal sums $X = \text{Ker}(T) \oplus \text{Ran}(T)$ and $X = \text{Ker}(S) \oplus \text{Ran}(S)$, we get the matrix decompositions

$$(3.6) \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T) \end{bmatrix}$$

and

$$(3.7) \quad S = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S) \end{bmatrix},$$

with respect to the orthogonal sums of submodules. Moreover, the adjoint of S is given by

$$(3.8) \quad S^* = \begin{bmatrix} S_1^* & S_2^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{Ran}(T) \\ \text{Ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Ran}(S) \\ \text{Ker}(S) \end{bmatrix}.$$

Since $T_1 : \text{Ran}(T) \rightarrow \text{Ran}(T)$ is invertible we obtain $\text{Ker}(TS) \cap \text{Ran}(S) = \text{Ker}(S_1)$, which implies that

$$(3.9) \quad \text{Ker}(TS) = \text{Ker}(TS) \cap (\text{Ker}(S) \oplus \text{Ran}(S)) = \text{Ker}(S) \oplus \text{Ker}(S_1).$$

Since TS is EP, $\text{Ran}(TS) = \text{Ran}(S^*T^*) \subseteq \text{Ran}(S^*) = \text{Ran}(S)$, which implies $\text{Ran}(TS) \subseteq \text{Ran}(T) \cap \text{Ran}(S)$.

For the converse, let $y \in \text{Ran}(T) \cap \text{Ran}(S)$ and $y \in \text{Ran}(TS)^\perp = \text{Ker}(TS) = \text{Ker}(S) \oplus \text{Ker}(S_1)$. Then $S_1 y = 0$. Using the matrix decompositions (3.7) and (3.8) for S and S^* ,

respectively, and fact that $Ran(S) = Ran(S^*)$, we obtain

$$(3.10) \quad \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} S_1^* & S_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix},$$

for some $z \in Ran(T)$ and $w \in Ker(T)$. Then $S_1y = S_2y = 0$, for every $y \in Ran(T) \cap Ran(S)$, that is,

$$\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $y \in Ker(S) \cap Ran(S) = \{0\}$ which yields $Ran(T) \cap Ran(S) \subseteq Ran(TS)$. That is, the implication (i) \rightarrow (iii) holds.

Suppose TS is an EP operator with closed range and $Ker(T) + Ker(S)$ is dense in its biorthogonal complement. Then $Ran(TS) = Ran(T) \cap Ran(S)$ by the preceding argument, which implies that

$$\begin{aligned} \overline{Ker(T) + Ker(S)} &= (Ker(T) + Ker(S))^{\perp\perp} = (Ran(T) \cap Ran(S))^{\perp} \\ &= Ran(TS)^{\perp} = Ker(TS). \end{aligned}$$

Conversely, suppose $Ran(TS) = Ran(T) \cap Ran(S)$ and $Ker(TS) = \overline{Ker(T) + Ker(S)}$ then $Ran(TS)$ is a closed submodule as an intersection of two closed submodules $Ran(T)$ and $Ran(S)$. Hence, $Ran((TS)^*)$ is orthogonally complemented in X , cf. [14, Theorem 3.2]. We find

$$\begin{aligned} Ran((TS)^*) &= Ker(TS)^{\perp} = \overline{Ker(T) + Ker(S)}^{\perp} \\ &= Ran(T) \cap Ran(S) = Ran(TS), \end{aligned}$$

i.e. TS is an EP operator with closed range. \square

Recall that a C^* -algebra of compact operators is a c_0 -direct sum of elementary C^* -algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i , $i \in I$, i.e. $\mathcal{A} = c_0\text{-}\bigoplus_{i \in I} \mathcal{K}(H_i)$, cf. [1, Theorem 1.4.5]. Suppose \mathcal{A} is an arbitrary C^* -algebra of compact operators. It is well known that every norm closed submodule of every Hilbert \mathcal{A} -module is automatically an orthogonal summand. Further generic properties of the category of Hilbert C^* -modules over C^* -algebras which characterize precisely the C^* -algebras of compact operators have been found in [8, 9, 10] and references therein. We can reformulate Theorem 3.7 in terms of bounded \mathcal{A} -linear maps on Hilbert C^* -modules over C^* -algebras of compact operators.

Corollary 3.8. *Suppose \mathcal{A} is an arbitrary C^* -algebra of compact operators, X is a Hilbert \mathcal{A} -module and $T, S \in \mathcal{L}(X)$ are EP operators with closed range. Then TS is an EP operator with closed range if and only if $Ran(TS) = Ran(T) \cap Ran(S)$ and $Ker(TS) = \overline{Ker(T) + Ker(S)}$.*

Koliha in [13] gave necessary and sufficient conditions for elements of C^* -algebras which commute with their Moore-Penrose inverse. He also studied conditions which ensure that the property is preserved under multiplication. As a special case of our results we recover some parts of Theorem 4.3 of [13].

Corollary 3.9. *Suppose \mathcal{A} is an arbitrary C^* -algebra and a, b and ab commute with their Moore-Penrose inverse. Then $ab\mathcal{A} = a\mathcal{A} \cap b\mathcal{A}$.*

Corollary 3.10. *Suppose \mathcal{A} is an arbitrary C^* -algebra of compact operators and a and b commute with their Moore-Penrose inverse. Then ab commutes with its Moore-Penrose inverse if and only if $ab\mathcal{A} = a\mathcal{A} \cap b\mathcal{A}$ and $(ab)_{-1} = \overline{a_{-1}(0) + a_{-1}(0)}$, in which $a_{-1}(0) = \{x \in \mathcal{A} : xa = 0\}$.*

Recall that every C^* -algebra is an \mathcal{A} -module on its own and define the bounded operators $L_a : \mathcal{A} \rightarrow \mathcal{A}$, $L_a(x) = ax$ then $a^{-1}(0) = \text{Ker}(L_a)$ and $\text{Ran}(L_a) = a\mathcal{A}$. The above facts follows from Theorem 3.7 and Corollary 3.8.

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